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STOCHASTIC FLOWS WITH STATIONARY DISTRIBUTION FOR TWO-DIMENSIONAL INVISCID FLUIDS

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We consider the Euler equation for an incompressible fluid in a general bounded domain of $\mathbb{R}^2$ with stochastic initial data. Extending previous work (for a fluid in a periodic box) we prove that the distribution of velocities u given as the standard normal distribution $\mu_{\beta,\gamma}$ with respect to the quadratic form $\gamma S(u)+\beta H(u)$, with $\beta, \gamma \geqslant 0$, S, H being respectively the entropy and energy, is infinitesimally invariant with respect to the dynamics given by the Euler equation, in the sense that there is a one parameter group of unitary operators in $L^2(\mu_{\beta,\gamma})$ with generator coinciding on a dense domain with the Liouville operator associated to the Euler flow. We also mention problems connected with proving the global invariance and the uniqueness of the stochastic flow.

Euler equation * stochastic flow * incompressible fluid * stochastic initial data

1. Introduction

There are situations in mathematics, physics and other sciences where some set of nonlinear partial differential equations is given, which should describe the evolution in time of a system, given suitable initial conditions. The initial conditions should be relevant to the problem at hand. It often happens that the set of relevant initial conditions for a given problem can only be described statistically by some initial distribution, and as a consequence also the time evolution is then studied naturally by looking at the time evolution of the initial distribution associated with the set of partial differential equations. A typical example is the statistical approach to hydrodynamics, where the partial differential equations for incompressible fluids are the Navier–Stokes equations. This statistical approach can be traced back to Reynolds and Taylor and has received a big impetus from Kolmogorov's work. Nowadays the stochastic approach is one of the main approaches to hydrodynamics, see e.g. [1, 6, 9, 27, 29].

The basic aims of a statistical approach in the above sense are to find stationary distributions (i.e. distributions invariant under the time evolution associated with

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the equations), to establish appropriate uniqueness results for them and give theorems about the asymptotic approach of the system towards such stationary distributions.

Problems of this type have been discussed mathematically, especially in connection with questions of the theory of stochastic processes and stochastic fields, the latter particularly stimulated by fundamental problems of statistical mechanics and dynamical systems, see e.g. [2, 3]. In the present paper we shall be concerned with the stochastic fields associated with plane flows of an incompressible perfect fluid, described classically by the Euler equations. The idea that to such a fluid a stationary distribution of Gaussian type should be associated has appeared repeatedly in the literature, one early reference being T.D. Lee [4]. The discussion was pursued quite extensively e.g. by K.H. Kraichnan [5] (see also e.g. [6]). For other discussions of the mathematical problems involved see [7], [8] and [9]. As was proven in [15] for almost all realizations with respect to such stationary measures the energy is infinite, so that certainly the corresponding fields are not classical velocity fields (known to exist in this plane case by e.g. [10–13]) so the statistical description here is really concerned with situations physically different to the one described by the classical equations. The expectation expressed in the physical literature mentioned above is naturally that such statistical stationary distributions really arise as limit distributions (relaxation to equilibrium). For the discussion of numerical evidence of this see e.g. [14]. The stationary distributions associated with the Euler flow also play a relevant reference role in the discussion of the random fields associated with the Navier–Stokes equations, see e.g. [1, 6, 8].

A first mathematical realization of the stationary Gaussian measures associated with the Euler equations in a plane box with periodic boundary conditions have been given by M.R. de Faria and ourselves [15] and by Boldrighini and Frigio [16] (see also the subsequent work [17]). In particular, it was shown in [15] that the measures $\mathrm{d}\mu_{\beta,\gamma}$ formally given for any constants $\beta \geqslant 0$, $\gamma > 0$ by

$$\mathrm{d}\mu_{\beta,\gamma}(\omega) = \text{const. } \mathrm{e}^{-\beta :H:_\gamma(\omega)}\, \mathrm{e}^{-\gamma S(\omega)}\, \mathrm{d}\omega,$$

where $\omega = (\omega_k)$ are the Fourier components of a realization of φ, φ being such that $(-\partial_2\varphi, \partial_1\varphi)$ is the velocity field, S is the entropy and $:H:_\gamma$ is the renormalized energy, exist and are infinitesimally invariant under the flow induced by the Euler equations, i.e. almost surely for the random fields given by $\mu_{\beta,\gamma}$ the Euler equations have a sense, infinitesimally in time (this will be explained below).

In the present paper we discuss the general case of a plane incompressible perfect fluid described classically by Euler equations in a bounded open region, with a not necessarily simply connected boundary. We first reduce the equations to equations for a scalar quantity φ (Section 2) and in Section 3 we give the mathematical definitions of measures of the type $\mathrm{d}\mu_{\beta,\gamma}$ for this case. We show in particular that the classical energy is almost surely infinite with respect to these measures. In Section 4 we prove the infinitesimal invariance of these measures, hence the infinitesimal stationarity of the associated stochastic fields. In Section 5 we show that, in the case of a simply connected domain, at least one flow exists which is defined by a strongly continuous unitary group in $L^2(\mathrm{d}\mu_{\beta,\gamma})$, given by a self-adjoint extension of the vector field B(Liouville operator) associated with the classical Euler equations. The problem of proving uniqueness of the flow (here left open) has an analogue in the classical statistical mechanics of infinitely many particles. (The latter has been solved for a class of one dimensional systems [18].)

We also mention a partial result from [24, 22, 23] towards the existence of a flow defined pointwise on the support of $\mu_{\beta,\gamma}$, as limit of the unique flow associated with the finite system of ordinary differential equations obtained from Euler equations by cutting off all energies larger than a fixed energy.

We also remark that the stochastic fields associated with the measures $\mathrm{d}\mu_{\beta,\gamma}$ are solutions of the stationary Hopf equation. The Hopf equation was introduced by E. Hopf in [19] as a functional equation intended to describe the time evolution of stochastic fields. For more recent discussions see [20]. That the stochastic fields associated with the measures $\mathrm{d}\mu_{\beta,\gamma}$ and other kind of measures ("Poissonian states"), in the case of the inviscid Euler flow with periodic boundary conditions in a square, satisfy the stationary Hopf equation was pointed out by Boldrighini and Frigio [16, 17]. Our present work extends the results to the general two-dimensional case.

Some of the results of this paper have been announced in [21, 22, 30, 31]. Let us finally remark that recent work by Benfatto, Picco and Pulvirenti [35] exhibits the Gaussian measures $\mu_{\beta,\gamma}$ as limit of the canonical Gibbs measures (Poissonian states) associated to a gas of vortices (see also [7–9, 17, 21, 22, 25–27, 31] for further work on the relations between two-dimensional vortex models, Sine-Gordon equation, Coulomb gas and Gaussian measures of the above type).

2. The Euler equation for an incompressible fluid in two dimensions

The Euler equation for an incompressible fluid in $\mathbb{R}^2$ is given by

$$\frac{\partial}{\partial t} u = -(u\cdot\nabla)u - \nabla p, \quad \operatorname{div} u = 0, \tag{2.1}$$

with $u=(u_1, u_2)$ the velocity field,

$$\nabla \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right), \qquad u\cdot\nabla \equiv u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2}.$$

The function $p(x)$ is called the pressure. The natural generalization of (2.1) to arbitrary domains $\Lambda\subset\mathbb{R}^2$ with piecewise C^1 boundary $\partial\Lambda$ is

$$\frac{\partial}{\partial t} u = -(u\cdot\nabla)u - f, \quad \operatorname{rot} f = 0, \quad \operatorname{div} u = 0 \tag{2.2}$$

with

$$\operatorname{rot} f \equiv -\frac{\partial}{\partial x_2} f_1 + \frac{\partial}{\partial x_1} f_2 .$$

Here $f=(f_1, f_2)$ is the force. In fact, when Λ is simply connected, in particular $\Lambda=\mathbb{R}^2$, then $\operatorname{rot} f=0$ iff $f=\nabla p$ for some p and thus, in this case, (2.2) coincides with (2.1). We shall study (2.2) with the boundary condition

$$n \cdot u=0 \quad \text{on } \partial\Lambda, \tag{2.3}$$

where u is the unit normal to $\partial\Lambda$. This expresses the fact that there should be no flux of the fluid through the boundary $\partial\Lambda$, i.e. the fluid is physically confined to Λ.

Now let ω be the one-form

$$\omega=u_2\,\mathrm{d}x_1-u_1\,\mathrm{d}x_2. \tag{2.4}$$

Then $\operatorname{div} u=0$ is equivalent with $\mathrm{d}\omega=0$. Thus $\operatorname{div} u=0$ implies that ω is a closed form. In this case for any closed curve C in Λ, $\int_C \omega$ only depends on the homology class of the curve C. Hence to evaluate $\int_C \omega$ it is enough to know the integrals $\int_{\partial^k\Lambda} \omega$, where $\partial^k\Lambda$, $k=1, \ldots, n$ are the connected components of $\partial\Lambda$. By (2.3) and (2.4) we have

$$\int_{\partial^k\Lambda} \omega=\int_{\partial^k\Lambda} n \cdot \mathrm{d}s=0, \tag{2.5}$$

with $\mathrm{d}s$ the length element of $\partial^k\Lambda$.

Hence ω is in the zero cohomology class and therefore there is a function φ on Λ such that $\omega=\mathrm{d}\varphi$, i.e.

$$u=(-\partial_2\varphi, \partial_1\varphi) \equiv \nabla^{\perp}\varphi, \quad \nabla^{\perp} \equiv(-\partial_2, \partial_1), \quad \partial_i \equiv \frac{\partial}{\partial x_i}, \quad i=1,2. \tag{2.6}$$

The Euler equation (2.2) then takes the form

$$\partial_t \nabla^{\perp}\varphi=-\sum_i (\nabla_i^{\perp}\varphi) \nabla_i \nabla^{\perp}\varphi-f, \tag{2.7}$$

with $\partial_t \equiv \partial/\partial t$, and ∇_i resp. $\nabla_i^{\perp}$ the i-th components of ∇ resp. $\nabla^{\perp}$. Since $\sum_i \nabla_i^{\perp}\nabla_i^{\perp}=\Delta$ and $\sum_i \nabla_i^{\perp}\nabla_i=0$ and $\nabla^{\perp} \cdot f=\operatorname{rot} f=0$, we may eliminate f from (2.7) and get

$$\partial_t \Delta\varphi=-\sum_{i,j} \nabla_j^{\perp}[(\nabla_i^{\perp}\varphi) \nabla_i \nabla_j^{\perp}\varphi], \tag{2.8}$$

which is obviously the same as

$$\partial_t \Delta\varphi=-\sum_{i,j} \nabla_j[(\nabla_i\varphi) \nabla_i^{\perp} \nabla_j\varphi]. \tag{2.9}$$

Using $\sum_i \nabla_i^{\perp}\nabla_i=0$ we get

$$\partial_t \Delta\varphi=-\sum_{i,j} \nabla_j \nabla_i^{\perp}(\nabla_i\varphi \cdot \nabla_j\varphi), \tag{2.10}$$

and using $\sum_i (\nabla_i\nabla_j\varphi)(\nabla_i^{\perp}\nabla_j\varphi)=0$ (since $\sum_i \nabla_i\nabla_i^{\perp}=0$) and $\sum_i \nabla_i^{\perp}\nabla_i\varphi=0$, we get that (2.10) takes the form

$$\partial_t \Delta\varphi=-\nabla\varphi \cdot \nabla^{\perp}\Delta\varphi \tag{2.11}$$

or

$$\partial_t \Delta\varphi=\nabla^{\perp}\varphi \cdot \nabla\Delta\varphi. \tag{2.12}$$

Remark. If we identify functions with two-forms in the natural way then (2.12) may be written as

$$\partial_t \Delta\varphi = \mathrm{d}\varphi \wedge \mathrm{d}\Delta\varphi. \tag{2.13}$$

Recalling that $u = \nabla^\perp \varphi$ and that $n \cdot u = 0$ on $\partial\Lambda$ we get that

$$t \cdot \nabla\varphi = 0 \quad \text{on } \partial\Lambda, \tag{2.13'}$$

where t is the unit vector tangent to $\partial\Lambda$. Thus the derivative of φ along $\partial\Lambda$ is zero, which implies that φ is a constant c_k on each component $\partial^k\Lambda$ of $\partial\Lambda$. Hence we have proven that for any classical solution u of (2.2) there is a function φ on Λ s.t. (2.6)–(2.13) hold. Suppose conversely that φ is a classical solution of (2.12). Define

$$f \equiv -\partial_t \nabla^\perp\varphi - \sum_i (\nabla_i^\perp \varphi)\nabla_i \nabla^\perp \varphi. \tag{2.14}$$

From (2.7) and (2.8) we then have that

$$\nabla^\perp \cdot f = \operatorname{rot} f = 0. \tag{2.15}$$

Hence in this case $u \equiv \nabla^\perp\varphi$ is a solution of the Euler equation (2.2), and moreover if φ is a constant on each component of $\partial\Lambda$ then $n \cdot u = 0$ on $\partial\Lambda$. Thus we have proven the following theorem.

Theorem 2.1. *Let Λ be a domain in $\mathbb{R}^2$ such that $\partial\Lambda$ is piecewise C^1. Then u is a smooth solution in Λ of the Euler equation*

$$\partial_t u = -(u \cdot \nabla)u - f, \quad \operatorname{rot} f = 0, \quad \operatorname{div} u = 0$$

with $n \cdot u = 0$ on $\partial\Lambda$, where n is a unit normal to $\partial\Lambda$ if and only if $u = \nabla^\perp\varphi$, where φ is a smooth solution in Λ of the equation

$$\partial_t \Delta\varphi = \nabla^\perp\varphi \cdot \nabla\Delta\varphi$$

such that φ is constant on each component of $\partial\Lambda$. f is given in terms of φ by

$$f = -\partial_t \nabla^\perp\varphi - \sum_i \nabla_i^\perp \varphi \nabla_i \nabla^\perp \varphi.$$

Remark. If φ is a solution of the equation in Theorem 2.1 then $\varphi + c$, for any constant c, is also a solution.

Remark. If $\partial^k\Lambda$, $k = 1, \ldots, m$ are the components of $\partial\Lambda$, the condition $n \cdot u = 0$ on $\partial\Lambda$ can be replaced by $\int_{\partial^k\Lambda} n \cdot u \, \mathrm{d}s = 0$, as we saw in (2.5). In view of Theorem 2.1 and the latter Remark we shall call equation (2.12) with boundary condition (2.13′) "*the Euler equation*". We shall now express the solution of the Euler equation in terms of a function satisfying zero boundary condition on $\partial\Lambda$ and functions harmonic in Λ. In fact we have the following result.

Theorem 2.2. *Let Λ be a bounded domain of $\mathbb{R}^2$ with piecewise C^1 boundary $\partial\Lambda$. Let $\partial^0\Lambda$ be the boundary of the unbounded components of $\mathbb{R}^2-\Lambda$, and let $\partial^1\Lambda,\dots,\partial^m\Lambda$ be the connected components of $\partial\Lambda-\partial^0\Lambda$. Let α_k be the harmonic function in Λ, independent of t, such that $\alpha_k=1$ on $\partial^k\Lambda$ and $\alpha_k=0$ on $\partial\Lambda-\partial^k\Lambda$, $k=1,\dots,m$. Then u is a smooth solution of the Euler equation in Λ*

$$\partial_t u=-(u\cdot\nabla)u-f,\quad \operatorname{rot} f=u,\quad \operatorname{div} u=0$$

with boundary condition $n\cdot u=0$ on $\partial\Lambda$ iff there exist m real constants $\beta_1,\dots,\beta_m$, such that, with $\alpha\equiv\sum_{k=1}^m\beta_k\alpha_k$ and φ a smooth solution of the equation $\partial_t\Delta\varphi=\nabla^\perp\varphi\cdot\nabla\Delta\varphi+\nabla^\perp\alpha\cdot\nabla\Delta\varphi$ with boundary condition $\varphi=0$ on $\partial\Lambda$, one has $u=\nabla^\perp(\varphi+\alpha)$.

Remark. If $\partial\Lambda$ is connected i.e. $\partial\Lambda=\partial^0\Lambda$, then $\alpha=0$ on $\partial^0\Lambda$ implies $\alpha\equiv0$. In this case the equation for φ is the one in Theorem 2.1 and $\varphi=0$ on $\partial\Lambda$.

Proof. Suppose u solves the Euler equation in Λ, with the given boundary conditions. Then if we set $u=\nabla^\perp(\varphi+\alpha)$ with $\alpha=\sum_{k=1}^m\beta_k\alpha_k$, α_k as in the theorem, then $n\cdot u=0$ on $\partial\Lambda$. Moreover, $\operatorname{div} u=0$ (since $\nabla\cdot\nabla^\perp=0$) and inserting the expression for u in terms of φ, α in the Euler equation we get, using $\partial_t\alpha=0$:

$$\partial_t\nabla^\perp\varphi=-[\nabla^\perp(\varphi+\alpha)\cdot\nabla]\nabla^\perp(\varphi+\alpha)-f,$$

from which it follows, applying $\nabla^\perp$ that

$$\partial_t\Delta\varphi=-\nabla^\perp[\nabla^\perp(\varphi+\alpha)\cdot\nabla]\nabla^\perp(\varphi+\alpha)-\nabla^\perp f, \tag{2.16}$$

and hence, using $\nabla^\perp\cdot f=0$,

$$\partial_t\Delta\varphi=-\nabla^\perp[\nabla^\perp(\varphi+\alpha)\cdot\nabla]\nabla^\perp(\varphi+\alpha). \tag{2.17}$$

As in the passage from (2.8) to (2.9) this is equivalent to

$$\partial_t\Delta\varphi=-\sum_{i,j}\nabla_j\nabla_i^\perp[\nabla_i(\varphi+\alpha)\cdot\nabla_j(\varphi+\alpha)]. \tag{2.18}$$

Using

$$\sum_i(\nabla_i\nabla_j(\varphi+\alpha))(\nabla_i^\perp\nabla_j(\varphi+\alpha))=0,$$

and

$$\sum_{i,j}\nabla_i^\perp\nabla_i(\varphi+\alpha)\nabla_j\nabla_j(\varphi+\alpha)=0.$$

this is equivalent to

$$\partial_t\Delta\varphi=-\nabla(\varphi+\alpha)\cdot\nabla^\perp\Delta(\varphi+\alpha)$$

i.e., using $\Delta\alpha=0$,

$$\partial_t\Delta\varphi=-\nabla\varphi\cdot\nabla^\perp\Delta\varphi-\nabla\alpha\cdot\nabla^\perp\Delta\varphi, \tag{2.19}$$

or equivalently,

$$\partial_t \Delta\varphi = \nabla^\perp\varphi \cdot \nabla\Delta\varphi + \nabla^\perp\alpha \cdot \nabla\Delta\varphi. \tag{2.19'}$$

The fact that if φ a solution of this equation also $\varphi + c$, for any constant c, is a solution, leading to the same value of u, allows us to choose $\varphi = 0$ on $\partial\Lambda$. Conversely, let α_k, β_k, α, β be as in the theorem, and set $u = \nabla^\perp(\varphi + \alpha)$. Defining

$$f \equiv -\partial_t \nabla^\perp(\varphi+\alpha) - \nabla^\perp(\varphi+\alpha)\cdot\nabla\nabla^\perp(\varphi+\alpha),$$

we get the Euler equation for u, and

$$\begin{aligned}\operatorname{rot} f &= -\nabla^\perp\partial_t\nabla^\perp(\varphi+\alpha) - \nabla^\perp\cdot\nabla^\perp(\varphi+\alpha)\cdot\nabla\nabla^\perp(\varphi+\alpha) = \\ &= \partial_t\Delta\varphi - \sum_i \nabla^\perp\nabla_i^\perp(\varphi+\alpha)\cdot\nabla_i\nabla^\perp(\varphi+\alpha).\end{aligned}$$

As we saw above in (2.17)-(2.19), the right hand side can be written as $\partial_t\Delta\varphi + \nabla\varphi\cdot\nabla^\perp\Delta\varphi + \nabla\alpha\cdot\nabla^\perp\Delta\varphi$, which is zero by assumption. Hence $\operatorname{rot} f = 0$. That $\operatorname{div} u = 0$ follows from $u = \nabla^\perp(\varphi+\alpha)$, using $\nabla\cdot\nabla^\perp = 0$. The boundary condition for u follows from the fact that $\varphi = 0$ and $\alpha = \beta_k = \text{constant}$ on $\partial^k\Lambda$. □

Remark. $\beta_1, \ldots, \beta_m$ are constants of motion for the flow of the fluid. To gain some understanding of the physical interpretation of these constants let us look at the following example.

Example. Let $\Lambda = \{x \in \mathbb{R}^2 \mid 1 < |x| < 2\}$ and set $\partial^1\Lambda \equiv \{x \mid |x| = 1\}$ and $\partial^0\Lambda \equiv \{x \mid |x| = 2\}$. Then $m = 1$. Solving the Dirichlet boundary value problems we get

$$\alpha_1(x) = 1 - (\ln 2)^{-1} \ln |x|.$$

Setting $\alpha(x) = \beta_1\alpha_1(x)$, $\varphi \equiv 0$, we have that a solution of the Euler equation is

$$u = \nabla^\perp\alpha(x) = -\beta_1(\ln 2)^{-1}|x|^{-2}(-x_2, x_1).$$

Of course this is a time independent (i.e. stationary) solution, which describes a circular motion with a speed proportional to $-\beta_1$ and falling off at $|x| \to \infty$ as $|x|^{-1}$.

Now let u be a continuous function in $L^2(\Lambda, \mathrm{d}x)$ and define the *energy* functional $H(u)$ by

$$H(u) \equiv \tfrac{1}{2}\int_\Lambda u^2\,\mathrm{d}x. \tag{2.20}$$

If $u = \nabla^\perp(\varphi+\alpha)$ with φ, α smooth and bounded we get, with $H(u) \equiv \tilde{H}(\varphi, \alpha)$,

$$\begin{aligned}\tilde{H}(\varphi,\alpha) &= \tfrac{1}{2}\int_\Lambda |\nabla^\perp(\varphi+\alpha)|^2\,\mathrm{d}x = \tfrac{1}{2}\int_\Lambda |\nabla^\perp\varphi|^2\,\mathrm{d}x + \\ &\quad + \int_\Lambda \nabla^\perp\varphi\cdot\nabla^\perp\alpha\,\mathrm{d}x + \tfrac{1}{2}\int_\Lambda |\nabla^\perp\alpha|^2\,\mathrm{d}x.\end{aligned} \tag{2.21}$$

If φ is zero on $\partial\Lambda$, as in Theorem 2.2, we then get, using that α is harmonic in Λ and using Green's formula,

$$\tilde{H}(\varphi, \alpha) = -\tfrac{1}{2}\int_\Lambda \varphi\Delta\varphi \, \mathrm{d}x + \tfrac{1}{2}\int_\Lambda \alpha n \cdot \nabla^\perp \alpha \, \mathrm{d}s.$$

But if we choose α as in Theorem 2.2, we have that α is constant on each component of $\partial\Lambda$, and hence

$$n \cdot \nabla^\perp \alpha = (t \cdot \nabla)\alpha = 0. \tag{2.22}$$

From (2.20)-(2.22) we obtain $H(u) = \tilde{H}(\varphi)$ with

$$\tilde{H}(\varphi) \equiv -\tfrac{1}{2}\int_\Lambda \varphi\Delta\varphi \, \mathrm{d}x. \tag{2.23}$$

Remark. $\{\psi, \varphi\} \to -\int_\Lambda \psi\Delta\varphi \, \mathrm{d}x$ is a symmetric form on the space of C^2 functions which are zero at $\partial\Lambda$.

We now have the following result.

Theorem 2.3. *Let u be a smooth solution of the Euler equation and let φ be a smooth solution of the equation given in Theorem 2.2. We suppose that either Λ is bounded or else u, φ and their derivatives are such that $H(u)$, $\tilde{H}(\varphi)$ and the integrals occurring in the proof all exist. Then $H(u) = \tilde{H}(\varphi)$, with*

$$H(u) = \tfrac{1}{2}\int_\Lambda u^2 \, \mathrm{d}x, \qquad \tilde{H}(\varphi) = -\tfrac{1}{2}\int_\Lambda \varphi\Delta\varphi \, \mathrm{d}x.$$

Moreover

$$\frac{\partial}{\partial t} H(u) = \frac{\partial}{\partial t}\tilde{H}(\varphi) = 0.$$

Remark. This result says that the energy H is a constant of motion.

Proof. We have already proven above that $H(u) = \tilde{H}(\varphi)$. Moreover we have, using the definition of $\tilde{H}(\varphi)$ and the equation for φ, with α as in Theorem 2.2, together with $\varphi = 0$ on $\partial\Lambda$ and the above remark:

$$\begin{aligned} \partial_t \tilde{H}(\varphi) &= -\int_\Lambda \varphi\nabla^\perp\varphi \cdot \nabla\Delta\varphi \, \mathrm{d}x - \int_\Lambda \varphi\nabla^\perp\alpha \cdot \nabla\Delta\varphi \, \mathrm{d}x = \\ &= \int_\Lambda \nabla\varphi \cdot \nabla^\perp\varphi\Delta\varphi \, \mathrm{d}x + \sum_{i,j}\int_\Lambda \nabla_j\varphi\nabla_i^\perp\alpha\nabla_i\nabla_j\varphi \, \mathrm{d}x, \end{aligned} \tag{2.24}$$

where we have integrated by parts. Using $\nabla \cdot \nabla^{\perp} = 0$ we then get that the right hand side is equal to

$$\sum_{i,j} \int_{\Lambda} \nabla_j \varphi \nabla_i^{\perp} \alpha \nabla_i \nabla_j \varphi \, dx = \sum_j \int_{\partial \Lambda} \nabla_j \varphi (n \cdot \nabla^{\perp} \alpha) \nabla_j \varphi \, dx$$

$$- \sum_{i,j} \int_{\Lambda} \nabla_i \nabla_j \varphi \nabla_i^{\perp} \alpha \nabla_j \varphi \, dx. \tag{2.25}$$

Since α is constant on each component of $\partial \Lambda$ we have $n \cdot \nabla^{\perp} \alpha = t \cdot \nabla \alpha = 0$, which inserted in (2.24), (2.25) yields

$$\partial_t \hat{H}(\varphi) = -\sum_{i,j} \int_{\Lambda} \nabla_i \nabla_j \varphi \nabla_i^{\perp} \alpha \nabla_j \varphi \, dx. \tag{2.26}$$

On the other hand the left hand side is equal to (2.25). Using the symmetry mentioned before the theorem we get that the term on the left of (2.25) is actually equal to the opposite of the term on the right of (2.26). This is only possible when $\partial_t \tilde{H}(\varphi) = 0$. □

We shall now see that there are other integrals of the motion.

Theorem 2.4. *Let u be a smooth solution of the Euler equation and let φ be a smooth solution of the equation in Theorem 2.2. We suppose either Λ bounded or else u, φ and their derivatives with boundedness properties such that $S(u)$, $\tilde{S}(\varphi)$ and all integrals in the proof below exist. Then*

$$S(u) \equiv \tfrac{1}{2} \int_{\Lambda} (\mathrm{rot}\, u)^2 \, dx = \tilde{S}(\varphi)$$

with

$$\tilde{S}(\varphi) \equiv \tfrac{1}{2} \int_{\Lambda} (\Delta \varphi)^2 \, dx.$$

Moreover, $\partial_t S(u) = \partial_t \tilde{S}(\varphi) = 0$.

Remark. The quantity $S(u) = \tilde{S}(\varphi)$ is called "enstrophy". The theorem says that the enstrophy is a constant of the motion.

Proof. By definition $S(u) = \frac{1}{2} \int_{\Lambda} (\mathrm{rot}\, u)^2 \, dx$. Using $u = \nabla^{\perp}(\varphi + \alpha)$ and $\Delta \alpha = 0$ we get

$$S(u) = \tilde{S}(\varphi), \quad \text{with } \tilde{S}(\varphi) \equiv \tfrac{1}{2} \int_{\Lambda} (\Delta \varphi)^2 \, dx. \tag{2.27}$$

If φ is a solution of the equation in Theorem 2.2 we get

$$\partial_t S(\varphi) = \int_{\Lambda} \Delta \varphi \cdot \nabla^{\perp} \varphi \cdot \nabla \Delta \varphi \, dx + \int_{\Lambda} \Delta \varphi \nabla^{\perp} \alpha \cdot \nabla \Delta \varphi \, dx. \tag{2.28}$$

Integration by parts gives that the first term on the right is equal to

$$-\int \varphi \nabla^{\perp} \Delta\varphi \cdot \nabla \Delta\varphi = 0, \tag{2.29}$$

since $\nabla^{\perp}\Delta\varphi \cdot \nabla\Delta\varphi = 0$. Thus from (2.28), (2.29),

$$\partial_t S(\varphi) = -\int_\Lambda \Delta\varphi \nabla^{\perp}\alpha \cdot \nabla\Delta\varphi \, \mathrm{d}x. \tag{2.30}$$

Comparing (2.30) and (2.28) we get $\partial_t S(\varphi) = 0$. □

In the case of domains Λ with $\partial\Lambda$ simply connected, we also have the following result.

Theorem 2.5. *Let Λ be a domain in $\mathbb{R}^2$ such that $\partial\Lambda$ is piecewise C^1 and connected. Let u resp. φ be a smooth solution of the Euler equation as in Theorem 2.2. Assume either Λ bounded or u, φ such that*

$$S_f(u) \equiv \int_\Lambda f(\mathrm{rot}\, u(x)) \, \mathrm{d}x \quad \textit{exist for all } f \in C(\mathbb{R})$$

and all integrals used in the proof also exist, for $f \in C^2(\mathbb{R})$. Then

$$S_f(u) = \tilde{S}_f(\varphi), \quad \textit{with } \tilde{S}_f(\varphi) \equiv \int_\Lambda f(\Delta\varphi(x)) \, \mathrm{d}x.$$

Moreover

$$\partial_t S_f(u) = \partial_t \tilde{S}_f(\varphi) = 0.$$

Remark. This result implies that $S(u) = \tilde{S}_f(\varphi)$ is a constant of motion. For Λ bounded (e.g.) we can easily extend this to all $f \in C(\mathbb{R})$ by approximation.

Proof. By the Remark following Theorem 2.2, for $\partial\Lambda$ connected, $\alpha = 0$ and φ satisfies the equation in Theorem 2.1.

Defining $S_f(u)$ resp. $\tilde{S}_f(\varphi)$ as in the Theorem we get from $u = \nabla^{\perp}\varphi$, $\mathrm{rot}\, u = \Delta\varphi$, hence $S_f(u) = \tilde{S}_f(\varphi)$. For $f \in C^2(\mathbb{R})$ we can compute

$$\partial_t \tilde{S}_f(\varphi) = \int_\Lambda f'(\Delta\varphi) \cdot \nabla^{\perp}\varphi \cdot \nabla\Delta\varphi \, \mathrm{d}x,$$

where we used the equation for $\partial_t \Delta\varphi$ given in Theorem 2.1. Using $\varphi = 0$ on $\partial\Lambda$ and a partial integration we get

$$\partial_t \tilde{S}_f(\varphi) = -\int_\Lambda f''(\Delta\varphi) \cdot \nabla^{\perp}\Delta\varphi \cdot \nabla\Delta\varphi \, \mathrm{d}x.$$

The right hand side vanishes, because of $\nabla^{\perp}\Delta\varphi \cdot \nabla\Delta\varphi = 0$, which proves $\partial_t \tilde{S}_f(\varphi) = 0$. □

We close this section with a new formulation of the Euler equation. Let for u, φ as in Theorem 2.2:

$$\eta(x) \equiv \operatorname{rot} u(x) = \Delta\varphi(x). \tag{2.31}$$

Set

$$B(\Delta\varphi) \equiv \nabla^{\perp}\varphi \cdot \nabla\Delta\varphi + \nabla^{\perp}\alpha \cdot \nabla\Delta\varphi. \tag{2.32}$$

Then we can write the equation for φ in Theorem 2.2 as

$$\partial_t \eta(x) = B(\eta)(x). \tag{2.33}$$

But, as remarked in (2.10), (2.12),

$$\nabla^{\perp}\varphi \cdot \nabla\Delta\varphi = -\sum_{i,j} \nabla_j \nabla_i^{\perp}(\nabla_i\varphi \nabla_j\varphi), \tag{2.34}$$

and thus

$$B(\Delta\varphi) = -\sum_{i,j} \nabla_j \nabla_i^{\perp}(\nabla_i\varphi\nabla_j\varphi) + \nabla^{\perp}\alpha \cdot \nabla\Delta\varphi. \tag{2.35}$$

For $\psi \in C_0^{\infty}(\Lambda)$ let us set

$$\langle \psi, B(\eta)\rangle = \langle \psi, B(\Delta\varphi)\rangle = \int_{\Lambda} \psi(x) B(\eta)(x)\,\mathrm{d}x$$

(in the sense of generalized functions). Then we have from (2.35), using $\nabla \cdot \nabla^{\perp} = 0$ and integration by parts,

$$\langle \psi, B(\Delta\varphi)\rangle = -\sum_{i,j} \int_{\Lambda} (\nabla_i^{\perp}\nabla_j\psi)(x)\nabla_i\varphi(x)\nabla_j\varphi(x) \\ -\langle \nabla\psi \cdot \nabla^{\perp}\alpha, \Delta\varphi\rangle. \tag{2.36}$$

By a partial integration we get that the term with the sum is equal to

$$\sum_{i,j} \int_{\Lambda} (\nabla_i^{\perp}\nabla_j\psi)(x)\varphi(x)\nabla_i\nabla_j\varphi(x)\,\mathrm{d}x. \tag{2.37}$$

Let $g(x, y)$ be the kernel in $L^2(\Lambda)$ of $(-\Delta)^{-1}$ and let

$$g_{ij}(x, y) = \frac{\partial^2}{\partial x_i\,\partial x_j} g(x, y).$$

Then

$$\int g(x, y)\eta(y)\,\mathrm{d}y = \varphi(x)$$

and

$$\int g_{ij}(x, y)\eta(y)\,\mathrm{d}y = \nabla_i\nabla_j\varphi(x).$$

Inserting this into (2.37) we get that (2.37) is equal to

$$\sum_{i,j}\int_{\Lambda}\iint (\nabla_i^{\perp}\nabla_j\psi)(x)g(x,y)g_{ij}(x,z)\eta(y)\eta(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{2.38}$$

Thus from (2.36)-(2.38),

$$\begin{aligned}&\langle\psi, B(\Delta\varphi)\rangle\\ &\quad=\sum_{i,j}\int_{\Lambda}\iint (\nabla_i^{\perp}\nabla_j\psi)(x)g(x,y)g_{ij}(x,z)\eta(y)\eta(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\qquad-\langle\nabla\psi\cdot\nabla^{\perp}\alpha, \Delta\varphi\rangle.\end{aligned} \tag{2.39}$$

Hence we have proven the following result.

Theorem 2.6. *Let u, φ, α be as in Theorem 2.2. Set $\eta=\operatorname{rot} u=\Delta\varphi$. Then*

$$\partial_t\eta=B(\eta),\quad \text{with}\quad B(\eta)=\nabla^{\perp}\varphi\cdot\nabla\Delta\varphi+\nabla^{\perp}\alpha\cdot\nabla\Delta\varphi.$$

Moreover, with $\psi\in C_0^{\infty}(\Lambda)$ and a pairing $\langle\cdot,\cdot\rangle$ in the sense of generalized functions

$$\begin{aligned}\int\frac{\partial}{\partial t}\langle\psi,\eta\rangle&=\langle\psi,B(\eta)\rangle\\ &=\sum_{i,j}\int_{\Lambda}\iint (\nabla_i^{\perp}\nabla_j\psi)(x)g(x,y)g_{ij}(y,z)\eta(y)\eta(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\quad-\langle\nabla\psi\cdot\nabla^{\perp}\alpha,\eta\rangle.\end{aligned}$$

Remark. From Theorem 2.6 we have that, as a linear functional on $C_0^{\infty}(\Lambda)$,

$$\begin{aligned}B(\eta)&=-\sum_{i,j}\int_{\Lambda}\iint \nabla_j\nabla_i^{\perp}g(\cdot,y)g_{ij}(\cdot,z)\eta(y)\eta(z)\,\mathrm{d}y\,\mathrm{d}z+\nabla^{\perp}\alpha\cdot\nabla\eta\\ &=-\sum_{i,j}\nabla_j\nabla_i^{\perp}((-\Delta)^{-1}\eta)(\cdot)(\nabla_i\nabla_j(-\Delta)^{-1}\eta)(\cdot)+\nabla^{\perp}\alpha\cdot\nabla\eta.\end{aligned} \tag{2.40}$$

For later use it is convenient to extend this linear functional to other functions η.

Let Λ be a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Lambda$ piecewise C^1, so that the Dirichlet Laplacian (i.e. the Laplacian with Dirichlet boundary conditions on $\partial\Lambda$) has discrete spectrum $0<\lambda_0\leq\lambda_1\leq\cdots$, with corresponding eigenfunctions φ_n so that $-\Delta\varphi_n=\lambda_n\varphi_n$. Let us consider functions $\tilde\eta$ on Λ of the form $\tilde\eta_N(x)=\sum_{\lambda_n\leq N}c_n\varphi_n(x)$ with c_n suitable constants, $c_n=0$ for $\lambda_n>N$. Define

$$B(\tilde\eta_N)\equiv-\sum_{i,j}\sum_{n,m}\lambda_n^{-1}c_n\lambda_m^{-1}(\nabla_j\nabla_i^{\perp}\varphi_n)(\nabla_i\nabla_j\varphi_m)+\sum_n c_n\nabla^{\perp}\alpha\cdot\nabla\varphi_n. \tag{2.41}$$

Using $\sum_i\nabla_i\nabla_j\nabla_i^{\perp}\varphi_n=0$ we get

$$\begin{aligned}B(\tilde\eta_N)&=\sum_{i,j}\sum_{n,m}\lambda_n^{-1}c_n\lambda_m^{-1}c_m\nabla_i\nabla_j(\nabla_i^{\perp}\varphi_m)(\nabla_j\varphi_n)\\ &\quad+\sum_n c_n\nabla^{\perp}\alpha\cdot\nabla\varphi_n.\end{aligned} \tag{2.42}$$

Hence we have for $\psi \in C_0^\infty(\Lambda)$, integrating by parts and observing that $\nabla^\perp\varphi_n\nabla\varphi_n = 0$, so that it is enough to sum over $n \neq m$:

$$\langle \psi, B(\tilde{\eta}_N)\rangle = -\sum_{n\neq m} \lambda_n^{-1} c_n \lambda_m^{-1} c_m \sum_{i,j} \langle \nabla_i \nabla_j \psi, \nabla_i^\perp \varphi_m \nabla_j \varphi_n\rangle$$
$$-\sum_n c_n \langle \nabla\psi \cdot \nabla^\perp \alpha, \varphi_n\rangle. \tag{2.43}$$

From (2.40) we also have

$$B(\tilde{\eta}_N) = -\sum_{i,j}\sum_{n,m} c_n c_m \int_\Lambda \iint \nabla_j \nabla_i^\perp g(\cdot, y) g_{ij}(\cdot, z) \varphi_n(y)\varphi_m(z)\,\mathrm{d}y\,\mathrm{d}z$$
$$+\sum_n c_n \nabla^\perp \alpha \cdot \nabla\varphi_n. \tag{2.44}$$

Hence, integrating by parts,

$$\langle \psi, B(\tilde{\eta}_N)\rangle$$
$$= \sum_{i,j}\sum_{n,m} c_n c_m \int_\Lambda \iint \nabla_i^\perp \nabla_j \psi(x) g(x,y) g_{ij}(x,z)\varphi_n(y)\varphi_m(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$-\sum c_n \langle \nabla\psi \cdot \nabla\alpha, \varphi_n\rangle. \tag{2.45}$$

Let us finally give the expressions of the enstrophy and energy functionals for u replaced by $\tilde{\eta}_N$. We have from the definition of the enstrophy functional in Theorem 2.4 that

$$S(\tilde{\eta}_N) = \tfrac{1}{2}\int_\Lambda \tilde{\eta}_N^2\,\mathrm{d}x = \tfrac{1}{2}\int_\Lambda \Big(\sum_n{}' \varphi_n y_n\Big)^2 \mathrm{d}x$$
$$= \tfrac{1}{2}\sum_{n,m} c_n c_m \int_\Lambda \varphi_n\varphi_m\,\mathrm{d}x = \tfrac{1}{2}\sum_n c_n^2 \int_\Lambda c_n^2 \int_\Lambda \varphi_n^2\,\mathrm{d}x = \tfrac{1}{2}\sum_n c_n^2, \tag{2.46}$$

where we used $\varphi_n \perp \varphi_m$ in $L^2(\Lambda)$ for $n \neq m$ and $\|\varphi_n\|_2 = 1$, and the notation $\sum_n'$ for the sum over all n such that $\lambda_n \leq N$. Moreover, using the definition of the energy functional in (2.20) we have, with $\tilde{U}_N$ s.t. rot $\tilde{U}_N = \tilde{\eta}_N$, using also $H(u) = \tilde{H}(\varphi)$ and

$$H(\tilde{u}_N) = \tfrac{1}{2}\int_\Lambda |\tilde{u}_N|^2\,\mathrm{d}x = -\tfrac{1}{2}\int_\Lambda \varphi_N \Delta\varphi_N\,\mathrm{d}x, \tag{2.47}$$

with φ_N such that

$$\Delta\varphi_N = \tilde{\eta}_N = \sum c_n\varphi_n. \tag{2.48}$$

From this we have

$$\varphi_N = (-\Delta)^{-1}\sum c_n\varphi_n = -\sum \lambda_n^{-1} c_n \varphi_n. \tag{2.49}$$

From (2.49) and (2.47) we get

$$H(\tilde{u}_N) = \tfrac{1}{2}\sum_{n,m} \lambda_n^{-1} c_n c_m \int \varphi_n\varphi_m\,\mathrm{d}x$$
$$= \tfrac{1}{2}\sum_n \lambda_n^{-1} c_n^2. \tag{2.50}$$

Remark. Define FC^1 to be the space of all functions $f(\eta)$ of the form

$$f(\eta)=\tilde{f}(\langle\varphi_{i_1},\eta\rangle,\ldots,\langle\varphi_{i_n},\eta\rangle),$$

for some $n, i_1,\ldots,i_n$, with $\tilde{f}\in C_b^1(\mathbb{R}^n)$. Define, for any $f\in FC^1$,

$$Bf(\eta)\equiv\sum_k B_k(\eta)\frac{\partial}{\partial y_k}f,$$

with $y_k\equiv\langle\varphi_k,\eta\rangle$. Then for η the solution of the Euler equation in Theorem 2.6

$$\frac{\partial}{\partial t}f(\eta(t))=\sum_k\frac{\partial}{\partial y_k}f(\eta(t))\frac{\partial y_k}{\partial t}=\sum_k\frac{\partial}{\partial y_k}f(\eta(t)).$$

On the other hand, for any smooth function g we get from the Euler equation for u,

$$\frac{\partial}{\partial t}u=-(u\cdot\nabla)u-f,$$

$$\frac{\partial}{\partial t}g(u(t))=-\int_\Lambda\frac{\delta g(u)}{\delta u_i(x)}((u\cdot\nabla u_i(x)+f(x))\,\mathrm{d}x,$$

where $\delta/\delta u_i(x)$ is the Gâteaux-derivative. Looking upon η as a function of u, using $\eta=\operatorname{rot}u$, we can consider $f(\eta)$ as a function $\hat{f}(u)$ and we obtain

$$\frac{\partial}{\partial t}f(u(t))=B\hat{f}(u(t))=-\int_\Lambda\frac{\delta\hat{f}(u)}{\delta u_i(x)}((u\cdot\nabla u_i(x)+f(x))\,\mathrm{d}x.$$

B is the "Liouville operator" associated with the Euler equation (cf. [18]). This equation will be useful in the discussion of the Hopf equation for hydrodynamics, in Section 5.

3. The Gibbs measure for the Euler flow and the random field $B(\eta)$

Let y_n, $n\in\mathbb{N}$ be a sequence of independent real-valued random variables, each of which is normally distributed with mean zero and variance γ^{-1}, for some $\gamma>0$. Then

$$\Pr(y_n\leqslant\lambda)=(\gamma/2\pi)^{1/2}\int_{-\infty}^{\lambda}\mathrm{e}^{-(\gamma/2)t^2}\,\mathrm{d}t,$$

where $\Pr(\cdot)$ means probability.

Let μ_γ be the probability law for the random vector $(y_n, n\in\mathbb{N})$, so that

$$\mu_\gamma(y_n\leqslant\lambda, y_m\in\mathbb{R}, m\neq n)=\Pr(y_n\leqslant\lambda).$$

μ_γ is a measure on $\mathbb{R}^{\mathbb{N}}$; in fact it is the standard normal distribution associated with the Hilbert space $l^2(\mathbb{N})$ with scalar product $(y,y')\equiv\gamma\sum y_n y'_n$, when $y=(y_n)$, $y'=(y'_n)$. It is well known that, if $\gamma\neq\gamma'$, then μ_γ and $\mu_{\gamma'}$ are orthogonal (cf. e.g. [32, Theorem 3.1]).

Let Λ be a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Lambda$ piecewise C^1. Let Δ be the Laplacian in Λ with Dirichlet boundary conditions on $\partial\Lambda$, and let φ_n, $n\in\mathbb{N}$ be the orthogonal base in $L_2(\Lambda)$ consisting of eigenfunctions φ_n of $-\Delta$ with eigenvalues $0<\lambda_0\leq\lambda_1\leq\cdots$. For $(y_n, n\in\mathbb{N}, \mu_\gamma)$ as above, let

$$\eta(x)\equiv\sum_n \varphi_n(x)y_n. \tag{3.1}$$

For any $\varphi\in L^2(\Lambda)$ we have that

$$\langle\varphi, \eta\rangle=\sum_n \langle\varphi, \varphi_n\rangle y_n, \tag{3.2}$$

with $\langle\varphi, \varphi_n\rangle\equiv\int_\Lambda \varphi_n(x)\varphi(x)\,\mathrm{d}x$, converges in $L^2(\mu_\gamma)$, since

$$\sum_n |\langle\varphi, \varphi_n\rangle|^2\equiv\|\varphi\|_2^2<\infty.$$

In fact

$$\|\langle\varphi, \eta\rangle\|_2^2\leq\gamma^{-2}\|\varphi\|_2^2. \tag{3.3}$$

Moreover $E_{\mu_\gamma}(\langle\varphi, \eta\rangle)=0$, where E_{μ_γ} is the expectation with respect to μ_γ, and we used $\langle\varphi, \eta\rangle\in L^2(\mu_\gamma)\subset L^1(\mu_\gamma)$ and $E_{\mu_\gamma}(y_n)=0$. Moreover,

$$E_{\mu_\gamma}(|\langle\varphi, \eta\rangle|^2)=\sum_n \gamma|\langle\varphi, \varphi_n\rangle|^2=\gamma\|\varphi\|_2^2,$$

with $\|\varphi\|_2^2\equiv\langle\varphi, \varphi\rangle$, as follows from $E_{\mu_\gamma}(y_n^2)=\gamma$, $E_{\mu_\gamma}(y_n, y_m)=0$ for $n\neq m$, and the Bessel-Parseval equality. Thus $\langle\varphi, \eta\rangle$ is a random variable, with underlying probability μ_γ, mean 0 and variance $\gamma\|\varphi\|_2^2$. The restriction of $\langle\cdot, \eta\rangle$ to $C_0^\infty(\Lambda)$ can thus be looked upon as the white noise random field, with Fourier transform

$$E(\mathrm{e}^{\mathrm{i}\langle\varphi,\eta\rangle})=\mathrm{e}^{-(2\gamma)^{-1}\|\varphi\|_2^2}.$$

Since $C_0^\infty(\Lambda)$ is a nuclear space, by Minlos Theorem we can look upon the random field $\langle\varphi, \eta\rangle$ as with probability measure supported by the dual $C_0^{\infty'}(\Lambda)\equiv\mathscr{D}'(\Lambda)$ of $C_0^\infty(\Lambda)\equiv\mathscr{D}(\Lambda)$. Thus we have proven the following result.

Lemma 3.1. *Let Λ be a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Lambda$ piecewise C^1. Let μ_γ be the Gaussian measure giving the joint distribution of the sequence of independent $N(0; \gamma^{-1})$ distributed real valued random variables y_n, $n\in\mathbb{N}$. Let $\eta(x)=\sum_n \varphi_n(x)y_n$, with φ_n the eigenfunctions of the Dirichlet Laplacian in Λ. Then for any $\varphi\in L^2(\Lambda)$, $\langle\varphi, \eta\rangle\equiv\sum_n \langle\varphi, \varphi_n\rangle y_n$, with $\langle\varphi, \varphi_n\rangle\equiv\int_\Lambda \varphi_n(x)\varphi(x)\,\mathrm{d}x$, is an element of $L^2(\mu_\gamma)$. $\langle\varphi, \eta\rangle$ is thus a random variable with respect to μ_γ, with mean zero and covariance $\gamma\langle\varphi, \varphi\rangle$. The restriction of $\langle\cdot, \eta\rangle$ to $C_0^\infty(\Lambda)$ can be identified with the white noise random field of strength γ with respect to the countable nuclear rigging $C_0^\infty(\Lambda)\subset L^2(\Lambda)\subset C_0^\infty(\Lambda)'$. μ_γ and $\mu_{\gamma'}$ are orthogonal for $\gamma\neq\gamma'$.* □

We shall prove in the next lemma that we may insert any element η in the support of μ_γ in the expression for the right hand side of Euler's equation given in Theorem 2.6.

Remark. If we set $\tilde{\eta}_N = \sum_n' \varphi_n y_n$ in (2.46) we get $S(\tilde{\eta}_N) = \frac{1}{2}\sum_n' y_n^2$. Formally $S(\eta)$ is the limit when $N \to \infty$ of $S(\tilde{\eta}_N)$ and the Gaussian measure μ_γ is heuristically given by

$$\exp[-\gamma S(\eta)] \prod_{x\in\Lambda} \mathrm{d}\eta(x) \Big/ \int \exp[-\gamma S(\eta)] \prod_{x\in\Lambda} \mathrm{d}\eta(x)$$

(since its covariance is the inverse of the "quadratic form" $\gamma S(\eta)$).

Lemma 3.2. *For any $\psi \in L^2(\Lambda)$, $\langle \psi, \eta\rangle \in L^2(\mu_\gamma)$ as given in Lemma 3.1 (with ψ replacing φ) define*

$$\langle \psi, B_N(\eta)\rangle \equiv -\sum_{m\neq n}' \lambda_m^{-1}\lambda_n^{-1} y_m y_n \sum_{i,j} \langle \nabla_i \nabla_j \psi, \nabla_i^\perp \varphi_m \nabla_j \varphi_n\rangle$$
$$-\langle \nabla\psi \cdot \nabla^\perp \alpha, \eta\rangle,$$

with α the function of Theorem 2.2 (harmonic in Λ, zero on $\partial^0\Lambda$ and equal to fixed arbitrary numbers β_k on $\partial^k\Lambda$), $\sum_{m\neq n}'$ being the sum over all $m \neq n$ with $\lambda_m, \lambda_n \leq N$.

Then $\langle \psi, B_N(\eta)\rangle \in L^2(\mu_\gamma)$ and the strong limit of $\langle \psi, B_N(\eta)\rangle$ as $N \to \infty$ exists in $L^2(\mu_\gamma)$. Call $\langle \psi, B(\eta)\rangle$ this limit. Then $E_{\mu_\gamma}(\langle \psi, B(\eta)\rangle) = 0$. In particular $\langle \varphi_n, B(\eta)\rangle \in L^2(\mu_\gamma)$, $E_{\mu_\gamma}(\langle \varphi_n, B(\eta)\rangle) = 0$, for all $n \in \mathbb{N}$. Moreover

$$\langle \varphi_n, B(\eta)\rangle = s\text{-}\lim_{N\to\infty} \Bigg\{ -\sum_{\substack{k\neq l\\ k\neq n\\ l\neq n}}' \lambda_k^{-1}\lambda_l^{-1} y_k y_l \sum_{i,j} \langle \nabla_i\nabla_j \varphi_n, \nabla_i^\perp \varphi_k \nabla_j \varphi_l\rangle$$
$$- \sum_{k\neq n}' y_k \langle \nabla\varphi_n \cdot \nabla^\perp \alpha, \varphi_k\rangle \Bigg\}.$$

In particular, $\langle \varphi_n, B(\eta)\rangle$ is independent of y_n.

Remark. $\langle \psi, B(\eta)\rangle$ is by construction equal to the expression on the right hand side of $\langle \psi, B_N(\eta)\rangle$, dropping the restriction $\max_k \lambda_k \leq N$ from the sum. On the other hand if we substituted in the expression for $B(\tilde{\eta}_N)$ given by (2.43) $\tilde{\eta}_N$ with $\tilde{\eta}_N = \sum_n' \varphi_n y_n$ (thus c_n is replaced by y_n), then we see that $\langle \psi, B(\tilde{\eta}_N)\rangle$ coincides with $\langle \psi, B_N(\eta)\rangle$. Thus

$$\langle \psi, B(\tilde{\eta}_N)\rangle = \langle \psi, B_N(\eta)\rangle$$
$$= -\sum_{m\neq n}' \lambda_m^{-1}\lambda_n^{-1} y_m y_n \sum_{i,j} \langle \nabla_i\nabla_j \psi, \nabla_i^\perp \varphi_m \nabla_j \varphi_n\rangle$$
$$-\langle \nabla\psi \cdot \nabla^\perp\alpha, \eta\rangle. \tag{3.4}$$

Proof of Lemma 3.2. From the definition of $\langle \psi, B_N(\eta)\rangle$ and the above Remark we have

$$\langle \psi, B_N(\eta)\rangle = \langle \psi, B_N(\tilde{\eta}_N)\rangle,$$

with $\langle\psi, B(\tilde{\eta}_N)\rangle$ given by (2.45), with c_n replaced by y_n, i.e.

$$\begin{aligned}\langle\psi, B_N(\eta)\rangle &= \langle\psi, B(\tilde{\eta}_N)\rangle \\ &= -\sum_n{}' y_n \langle\nabla\psi\cdot\nabla\alpha, \varphi_n\rangle \\ &\quad + \sum_{i,j}\sum_{m,n}{}' y_m y_n \int_\Lambda \iint \nabla_i^\perp\nabla_j\psi(x) g(x,y) g_{ij}(x,z)\varphi(y)\varphi_m(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \qquad (3.5)\end{aligned}$$

Using the Schwarz inequality, the normalization $\|\varphi\|_2=1$ and the fact that $E(y_n y_m y_{n'} y_{m'}) \le C\gamma^{-2}$, we see that

$$E(|\langle\psi, B(\tilde{\eta}_N)\rangle|^2) \le C\gamma^{-2}\|A\|_2^2 + \|\nabla\psi\cdot\nabla^\perp\alpha\|_2^2,$$

where $\|A\|_2$ is the Hilbert-Schmidt norm of the operator

$$A \equiv \sum_{i,j}(-\Delta)^{-1}\psi_{ij}\nabla_i\nabla_j(-\Delta)^{-1}, \tag{3.6}$$

with ψ_{ij} the multiplication operator on $L^2(\Lambda)$ given by $\psi_{ij}(x) \equiv \nabla_i^\perp\nabla_j\psi(x)$. The kernel $A(x,y)$ of A is

$$A(x,y) \equiv \sum_{i,j}\int_\Lambda g(x,z)(\nabla_i^\perp\nabla_j\psi)(z) g_{ij}(z,y)\,\mathrm{d}z. \tag{3.7}$$

This is of course a useful bound only if A belongs to the Hilbert-Schmidt class, which however is easy to prove, observing that

$$\|A\|_2 \le \sum_{i,j}\|\psi_{ij}\|\,\|\nabla_i\nabla_j(-\Delta)^{-1}\|\,\|(-\Delta)^{-1}\|_2, \tag{3.8}$$

where $\|\cdot\|$ denotes the operator norm. However $\|(-\Delta)^{-1}\|_2<\infty$, since $g(x,y)$ is continuous for $x\neq y$ and

$$g(x,y) = -\frac{1}{2\pi}\ln|x-y| + \mathrm{O}(|x-y|) \tag{3.9}$$

for $x-y\to 0$. Moreover $\|\psi_{ij}\| = \|\psi_{ij}\|_\infty < \infty$ (where $\|\cdot\|_\infty$ is the $L^\infty(\mathrm{d}x)$-norm) and $\|\nabla_i\nabla_j(-\Delta)^{-1}\| \le 1$.

Moreover, since $\|\nabla\psi\cdot\nabla^\perp\alpha\|_2^2<\infty$, by the fact that $\psi\in C_0^\infty(\Lambda)$, we thus have

$$\langle\psi, B(\tilde{\eta}_N))\rangle \in L^2(\mu_\gamma). \tag{3.10}$$

Furthermore, since $E(y_n y_m)=0$ for $n\neq m$, we get from (3.4) that

$$E(\langle\psi, B(\tilde{\eta}_N)\rangle) = 0. \tag{3.11}$$

Let us now prove that $\{\langle\psi, B(\tilde{\eta}_N)\rangle, N\in\mathbb{N}\}$ is a Cauchy sequence in $L^2(\mu_\gamma)$. We have for any $k\in\mathbb{N}$, from (3.5), (3.6), (3.7),

$$|\langle\psi, B(\tilde{\eta}_{N+k})\rangle - \langle\psi, B(\tilde{\eta}_N)\rangle|^2 = \left|\iiint A(y,z)\sum_{m,n}^{\sim} y_n\varphi_n(y) y_m\varphi_m(z)\,\mathrm{d}y\,\mathrm{d}z\right|^2,$$

with $\tilde{\sum}_{m,n}$ the sum over all m, n such that $N+1 \leq \lambda_m$, $\lambda_n \leq N+k$. Thus

$$E(|\langle \psi, B(\tilde{\eta}_{N+k})\rangle - \langle \psi, B(\tilde{\eta}_N)\rangle|^2)$$
$$= \tilde{\sum_{m,n}} \tilde{\sum_{m',n'}} E(y_m y_n y_{m'} y_{n'}) \iiint A(y,z) A(y',z')$$
$$\times \varphi_n(y)\varphi_{n'}(y')\varphi_m(z)\varphi_{m'}(z')\,\mathrm{d}y\,\mathrm{d}y'\,\mathrm{d}z\,\mathrm{d}z'.$$

We have $E(y_n y_{n'} y_m y_{m'}) = 0$ unless $n = n'$, $m = m'$ or $n = m$, $n' = m'$ or $n = m'$, $n' = m$. The term $n = n'$ is bounded by

$$E\left(\left|\left\langle A, \tilde{\sum_{m,n}} y_n \varphi_n y_m \varphi_m \right\rangle\right|^2\right) < \varepsilon,$$

whenever $N \geq N_0(\varepsilon)$, since

$$E\left[\left\langle A, \sum_{n,m} y_n \varphi_n y_m \varphi_m \right\rangle\right] \leq C\gamma^{-2} \sum_n \langle A, \varphi_n \varphi_m \rangle$$
$$\leq C\gamma^{-2} \|A\|_2,$$

where we used $\|\varphi_n\|_2 = 1$. The other terms are bounded in a similar way. This then proves that $\langle \psi, B(\tilde{\eta}_N)\rangle$ is an $L^2(\mu_\gamma)$-Cauchy sequence. Hence $\langle \psi, B(\tilde{\eta}_N)\rangle$ converges in $L^2(\mu_\gamma)$ to an element $\langle \psi, B(\eta)\rangle$ hence also in $L^1(\mu_\gamma)$. From (3.11) we then obtain $E(\langle \psi, B(\eta)\rangle) = 0$. The rest follows taking $\psi = \varphi_n$ and using (3.4). $\square$

We summarize the results of Lemma 3.1, 3.2 in the following theorem.

Theorem 3.3. *Let Λ be a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Lambda$, piecewise C^1. Let μ_γ be the white noise field of strength γ of Lemma 3.1. Then for $\eta = \sum_n \varphi_n y_n$, with φ_n the orthonormal eigenfunction of the Dirichlet Laplacian in Λ and y_n i.i.d. centered Gaussians with mean 0 and variance γ^{-1} we have $\langle \psi, \eta \rangle \in L^2(\mu_\gamma)$ for any $\psi \in L^2(\Lambda)$. Moreover for any $\psi \in C_0^\infty(\Lambda)$, $\langle \psi, B(\eta)\rangle$ exists as the $L^2(\mu_\gamma)$-limit of*

$$\langle \psi, B_N(\eta)\rangle = -\sum_{m \neq n}{}' \lambda_m^{-1} \lambda_n^{-1} y_m y_n \sum_{i,j} \langle \nabla_i \nabla_j \psi, \nabla_i^\perp \varphi_m \nabla_j \varphi_n \rangle$$
$$- \langle \nabla \psi \cdot \nabla^\perp \alpha, \eta \rangle,$$

for any α as in Lemma 3.2. Moreover $E(\langle \psi, B(\eta)\rangle) = 0$. $\langle \varphi_n, B(\eta)\rangle$ is independent of y_n. $\square$

Remark. For later use we remark that if we call P_m the closure in $L^2(\mu_\gamma)$ of the polynomials in y_n of degree smaller or equal to m, and set

$$P = \bigcup_m P_m, \tag{3.12}$$

then a simple verification shows $P \subset L^p(\mu_\gamma)$, for all $p > 0$. Moreover we have $B_n(\eta) \in P_2$.

4. The renormalized energy

We shall convince ourselves in a while that if we heuristically insert into the expression for the energy functional $H(u)$ a stochastic element u s.t. rot $u=\eta$ with η as in Theorem 3.3, then $H(u)$ has infinite $L^2(\mu_\gamma)$-norm. In fact let us take first, instead of such u's, rather $\tilde{u}_N$ as in (2.47)-(2.50). In this case by (2.50),

$$H(\tilde{u}_N)=\tfrac{1}{2}\sum_n{}' \lambda_n^{-1}y_n^2 \quad \left(\sum_n{}' \equiv \sum_{n,\lambda_n\leq N}\right). \tag{4.1}$$

Let us also consider the corresponding "cut-off renormalized energy functional" ("Wick ordered energy")

$$:H(\tilde{u}_N):\equiv H(\tilde{u}_N)-\tfrac{1}{2}\sum_n{}' \lambda_n^{-1}\gamma^{-1}, \tag{4.2}$$

so that

$$:H(\tilde{u}_N):=\tfrac{1}{2}\sum_n{}' \lambda_n^{-1}(y_n^2-\gamma^{-1}). \tag{4.3}$$

We remark that the correction in passing (4.1) to (4.3) is by a constant, such that

$$E(:H(\tilde{u}_N):)=0 \text{ (with } E\equiv\text{expectation resp. } \mu_\gamma). \tag{4.4}$$

In fact we have $E(y_n^2)=\gamma^{-1}$, so that each term in the sum (4.3) has mean zero. Also $:H(\tilde{u}_N):\in L^2(\mu_\gamma)$ for all N. Moreover $:H(\tilde{u}_N):$ is decreasing in γ. For $\gamma'<\gamma$ we have, writing $:H(\tilde{u}_N):_\gamma$ to underly the dependence on γ,

$$:H(\tilde{u}_N):_\gamma - :H(\tilde{u}_N):_{\gamma'}=-\tfrac{1}{2}\sum_n{}' \lambda_n^{-1}(\gamma'^{-1}-\gamma^{-1})<0. \tag{4.5}$$

We now remark that $\sum_{n=1}^\infty \lambda_n^{-1}$ diverges and $\sum_{n=1}^\infty \lambda_n^{-2}$ converges, since by Weyl's formula $\lambda_n\sim|\Lambda|^{-1}4\pi n$, where $|\Lambda|$ is the Lebesgue measure of Λ. We shall, however, show that $:H(\tilde{u}_N):$ converges in $L^2(\mu_\gamma)$ as $N\to\infty$. To see this we remark that for any $\varepsilon>0$ we have

$$\begin{aligned}\int\left|\sum_{n=m}^{m+l}\lambda_n^{-1}(y_n^2-\gamma^{-1})\right|^2 \mathrm{d}\mu_\gamma &= \sum_{n=m}^{m+l}\lambda_n^{-1}\lambda_m^{-1}\int(y_n^2-\gamma^{-1})(y_m^2-\gamma^{-1})\,\mathrm{d}\mu_\gamma\\ &=\sum_{n=m}^{m+l}\lambda_n^{-2}[2\gamma^{-2}-\gamma^{-2}-\gamma^{-2}+\gamma^{-2}]\\ &=\gamma^{-2}\sum_{n=m}^{m+l}\lambda_n^{-2}<\varepsilon,\end{aligned} \tag{4.6}$$

whenever $m\geq m_0$ for some $m_0=m_0(\varepsilon)$, since $\sum_n \lambda_n^{-2}<\infty$. This proves that $\sum_{n=1}^m \lambda_n^{-1}(y_n^2-\gamma^{-1})$ is a Cauchy sequence in $L^2(\mu_\gamma)$. Let us call $:H:(u)$ the limit element, so that

$$:H:(u)=s-\lim :H(\tilde{u}_N):. \tag{4.7}$$

Remark. The notation $:H:(u)$ should be understood with u given by η, $\eta \in \operatorname{supp} \mu_\gamma$. Let us consider $\|:H:\|_2$. From the fact that $:H(\tilde{u}_N):$ converges strongly in $L^2(\mu_\gamma)$ to $:H:(u)$, we get $\|:H:\|_2 = \lim_{N\to\infty} \|:H(\tilde{u}_N):\|_2$. But, similarly to (4.4), $\|:H(\tilde{u}_N):\|_2^2 = \gamma^{-2} \sum_n' \lambda_n^{-2}$ converges as $N \to \infty$ to

$$\gamma^{-2} \sum_n \lambda_n^{-2} = \gamma^{-2} \int_\Lambda \int_\Lambda |g(x, y)|^2 \, dx \, dy < \infty.$$

Since $:H: \in L^2(\mu_\gamma)$, we have $|:H:| < \infty$ μ_γ-a.s. This, however, implies that $\sum_n \lambda_n^{-1} y_n^2$ diverges μ_γ-a.s., otherwise we would have a limit element h in $L^2(\mu_\gamma)$. Then we would have

$$:H(\tilde{u}_N): + \sum_n{}' \gamma^{-1} \lambda_n^{-1} = \sum_n{}' \lambda_n^{-1} y_n^2 \to h$$

strongly in $L^2(\mu_\gamma)$. But then on the one hand s-lim $:H(\tilde{u}_N): = :H:(u)$, and on the other hand $:H:(u) = h - \lim_{N\to\infty} \sum_n' \gamma^{-1} \lambda_n^{-1} = +\infty$. Since both $:H(\tilde{u}_N):$ and h are finite μ_γ-a.s. but $\lim_{N\to\infty} \sum_n' \gamma^{-1} y_n^{-1} = +\infty$, we get a contradiction.

Thus the evalution of the unrenormalized energy functional H of Theorem 2.1 for u s.t. rot $u = \eta$ yields $\sum_n \lambda_n^{-1} y_n^2 = +\infty$ μ_γ-a.s. Finally we also have from (4.5) and the divergence of the r.h.s. of (4.5) as $N \to \infty$ that $:H:_\gamma - :H:_{\gamma'}$ is negative infinite μ_γ-a.e., where we denote by $:H:_\gamma$ the $L^2(\mu_\gamma)$-function constructed above, the index γ indicating the dependence on γ. Hence we have proven the following result.

Theorem 4.1. *Let $\tilde{\eta}_N = \sum_{\lambda_n \le N} \varphi_n y_n$, with y_n, φ_n as in Lemma 4.1. Then the energy*

$$H(\tilde{u}_N) = \tfrac{1}{2} \sum_{\lambda_n \le N} \lambda_n^{-1} y_n^2$$

associated with $\tilde{\eta}_N$ is in $L^2(\mu_\gamma)$, but diverges μ_γ-a.s. as $N \to \infty$. The renormalized energy

$$:H(\tilde{u}_N): \equiv H(\tilde{u}_N) - \sum_{\lambda_n \le N} \gamma^{-1} \lambda_n^{-1} = \tfrac{1}{2} \sum_{\lambda_n \le N} \lambda_n^{-1} (y_n^2 - \gamma^{-1})$$

is in $L^2(\mu_\gamma)$, has norm $\gamma^{-2} \sum_{\lambda_n \le N} \lambda_n^{-2}$, mean zero and converges strongly in $L^2(\mu_\gamma)$ to an element $:H:(u)$ of $L^2(\mu_\gamma)$.

In fact $:H: \in P_2$, in the notation explained in the remark at the end of Section 3. We have

$$\|:H:\|_2^2 = \gamma^{-2} \sum_n \lambda_n^{-2} = \gamma^{-2} \int_\Lambda \int_\Lambda |g(x, y)|^2 \, dx \, dy.$$

Moreover $E(:H:) = 0$. $:H:$ is a μ_γ-a.s. pointwise decreasing function of γ such that if $\gamma' < \gamma$ then $:H:_\gamma - :H:_{\gamma'}$ is negative infinite for μ_γ-a.e. η and for $\mu_{\gamma'}$-almost all η.

Remark. $:H:$ is a $L^2(\mu_\gamma)$-function, therefore a function of $\eta \in \operatorname{supp} \mu_\gamma$. We write the argument u for $:H:$ to remind us to the actual dependence of the corresponding (unnormalized) energy $H(u)$ in the case of smooth u (where then η is rot u).

Theorem 4.2. *For any $\beta \geq 0$, we have* $\mathrm{e}^{-\beta:H:} \in L^1(\mu_\gamma)$, *so that*

$$\mathrm{d}\mu_{\beta,\gamma} \equiv \left(\int \mathrm{e}^{-\beta:H:}\, \mathrm{d}\mu_\gamma \right)^{-1} \mathrm{e}^{-\beta:H:}\, \mathrm{d}\mu_\gamma$$

is a probability measure absolutely continuous with respect to μ_γ. *For all* $\beta \geq 0$, $\gamma > 0$, *we have* $:H: \in L^2(\mu_{\beta,\gamma})$ *and for any* $\psi \in C_0^\infty(\Lambda)$ *we have*

$$\langle \psi, B(\eta) \rangle \in L^2(\mu_{\beta,\gamma}).$$

Remark. The probability measures $\mu_{\beta,\gamma}$ can be looked upon heuristically as measures with density $\exp(-\beta H(u) - \gamma S(\eta))$ with respect to the heuristic infinite dimensional Lebesgue measure on the η. It is naturally therefore to call them the "*pure Gibbs measures for the Euler flow*".

The proof follows easily using $\sum_n \lambda_n^{-2} < \infty$, together with the fact that $\mathrm{d}\mu_\gamma$ and

$$\exp\left[-\sum_{j=1}^{k} \lambda_j^{-1}(y_j^2 - \gamma^{-1}) \right] \mathrm{d}\mu_\gamma \bigg/ \int \exp\left[-\sum_{j=1}^{k} \lambda_j^{-1}(y_j^2 - \gamma^{-1}) \right] \mathrm{d}\mu_\gamma$$

are Gaussian product measures and one can apply Kakutani's general results on equivalence of Gaussian measures.[1]

5. The infinitesimal invariance of the Gibbs measures

The enstrophy S as a functional of the classical solutions of the Euler equation is a conserved quantity (by Theorem 2.4). We shall thus say that $S(u)$ is invariant under the classical Euler flow $u(0) \to u(t)$ given by Theorem 2.4. Formally μ_γ is described by a density $\mathrm{e}^{-\gamma S}$ as remarked after Lemma 3.1. Would one be able to describe a flow on $C_0^\infty(\Lambda)'$ associated with the Euler equations (with initial conditions on $C_0^\infty(\Lambda)'$!), then one might hope to be able to prove that μ_γ is invariant under this flow, expecting the enstrophy to remain invariant. However the solutions of the Euler equation with such generalized initial conditions are not available (see however [15]-[17], [22]-[24], for steps in these directions). We also remark that μ_γ has support on velocity fields u s.t. rot $u = \eta$ for which the energy, as we saw in Theorem 4.1, is infinite μ_γ-a.s. So we cannot hope to be able to use results on existence of solutions of the Euler equation with initial conditions of finite energy.

We are however going to see that there is a natural concept of "infinitesimal invariance", referring to any possible "generalized Euler flow", and that μ_γ is precisely infinitesimal invariant.

Let (as in Section 2) FC^1 be the space of cylinder (tame) functions on $\mathscr{D}'(\Lambda) \equiv C_0^\infty(\Lambda)'$ which depend only on a finite number of coordinates $\eta_n \equiv \langle \varphi_n, \eta \rangle$, $\eta \in \mathscr{D}'(\Lambda)$,

[1] $\beta \geq 0$ in Theorem 4.2 can be relaxed to $\beta > -\lambda_1 \gamma$, see [31], which also contains new results on the supports of the measures $\mu_{\beta,\gamma}$ and the Poissonian states mentioned at the end of the introduction.

$n \in \mathbb{N}$. Thus $f \in \mathrm{FC}^1$ iff $f(\eta) = \tilde{f}(\langle \varphi_{n_1}, \eta \rangle, \ldots, \langle \varphi_{n_k}, \eta \rangle)$ for some $k \in \mathbb{N}$, some $n_i \in \mathbb{N}$, some $\tilde{f} \in C_b^1(\mathbb{R}^k)$. The φ_n are eigenfunctions of the Dirichlet Laplacian in the bounded domain Λ with C^1 boundary, as in Sections 2, 3, 4. Remarking that $\eta = \sum \varphi_n y_n$ so that $\langle \varphi_n, \eta \rangle = y_n$, we shall also use the notation $f(\eta) = \tilde{f}(y_{n_1}, \ldots, y_{n_k})$. Moreover we shall denote the derivative of $\tilde{f}(x_1, \ldots, x_k)$ with respect to its variable x_i by $\partial f / \partial x_i$.

It is easily verified (by a Stone–Weierstrass-theorem) that FC^1 is dense in $L^2(\mu_\gamma)$. For any $\eta \in \operatorname{supp} \mu_\gamma$, let

$$B_n(\eta) \equiv \langle \varphi_n, B(\eta) \rangle. \tag{5.1}$$

We saw in Theorem 3.3 that $B_n \in L^2(\mu_\gamma)$ and

$$B_n(\eta) \equiv \underset{N \to \infty}{s\text{-}\lim} \langle \varphi_n, B(\tilde{\eta}_N) \rangle = \underset{N \to \infty}{s\text{-}\lim} \langle \varphi_n, B_N(\eta) \rangle, \tag{5.2}$$

with $B(\tilde{\eta}_N)$ given by (3.4), (3.5). Let us define for $f \in \mathrm{FC}^1$ as above:

$$(Bf)(\eta) \equiv \sum_n B_n(\eta) \frac{\partial}{\partial y_n} f(\eta). \tag{5.3}$$

The sum on the right hand side being finite, and B_n being in $L^2(\mu_\gamma)$, $(\partial / \partial y_n) f$ in $L^\infty(\mu_\gamma)$, B is well defined or FC^1, and $Bf \in L^2(\mu_\gamma)$. Hence B is a linear densely defined operator on $L^2(\mu_\gamma)$. It might be useful to think of B as vector field on $\mathscr{D}'(\Lambda)$, with coefficients $B_n(\eta)$. We shall call B *the* (*local*) *infinitesimal generator of a generalized Euler flow.*

Remark. For classical solutions of the Euler equation, as remarked in (2.33), we have

$$\frac{\partial}{\partial t} \eta = B(\eta),$$

hence

$$\frac{\partial}{\partial t} \langle \varphi_n, \eta \rangle = B_n(\eta), \quad \text{i.e.} \ \frac{\partial}{\partial t} y_n = B_n(\eta),$$

using, also in this case, the notation $y_n \equiv \langle \varphi_n, \eta \rangle$. This implies, for $f \in \mathrm{FC}^1$ and for such classical u, that

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \sum_n \frac{\partial f}{\partial y_n} \frac{\partial}{\partial t} y_n = \sum_n \frac{\partial f}{\partial y_n} B_n(\eta) = (Bf)(\eta), \tag{5.4}$$

defining Bf also in this case by (5.3). In this case B is indeed the vector field (infinitesimal generator, "Liouville operator") corresponding to the (classical) Euler equation.

B is divergence free as seen from (5.3), since B_n is independent of y_n, i.e. $(\partial / \partial y_n) B_n = 0$, i.e. $\operatorname{div} B \equiv \sum_n (\partial B_n / \partial y_n) = 0$ on FC^1. It is therefore interesting to study the properties of the vector field B also for η which are not classical solutions of the Euler equation, namely those in the support of μ_γ. From Theorem 3.3 we have also in this case that B_n is independent of y_n, i.e.

$$\frac{\partial}{\partial y_n} B_n(\eta) = 0, \quad \mu_\gamma\text{-a.s.} \tag{5.5}$$

Hence div $B \equiv \sum (\partial B_n/\partial y_n)$ is zero on FC^1, μ_γ-a.s. We shall call a positive bounded measure μ on $\mathscr{D}'(\Lambda)$ *infinitesimally invariant under a generalized Euler flow* if

$$\int_{\mathscr{D}'(\Lambda)} Bf \,\mathrm{d}\mu = 0 \tag{5.6}$$

for all $f \in \mathrm{FC}^1$. This is clearly equivalent to $B^*1 = 0$, where B^* is the adjoint of B in $L^2(\mu)$ and 1 is the function identically one on $\mathscr{D}'(\Lambda)$.

Remark. If there exist solutions of the Euler equations $(\partial/\partial t)\eta = B(\eta)$ with initial conditions $\eta(0) \in \operatorname{supp} \mu_\gamma$ leaving μ_γ invariant (in the sense that the map $\eta(0) \to \eta(t)$ is a measure preserving map on $(\mathscr{D}'(\Lambda), \mu_\gamma))$, then we would have (5.4) and hence, integrating with respect to $\mathrm{d}\mu_\gamma$, $\int (\mathrm{d}/\mathrm{d}t) f \,\mathrm{d}\mu_\gamma = \int Bf \,\mathrm{d}\mu_\gamma$. By the invariance of μ_γ this would yield indeed (5.6).

In this sense the above denomination of "infinitesimal invariance" has some justification.

We shall now prove the following theorem.

Theorem 5.1. *Let $\mu_{\beta,\gamma}$ be the Gibbsian measures of Theorem* 4.2. *Let B be the densely defined operator on $L^2(\mu_{\beta,\gamma})$ with domain $D(B) = \mathrm{FC}^1$ given by*

$$Bf(\eta) = \sum_n B_n(\eta) \frac{\partial}{\partial y_n} f(\eta), \quad \eta \in \operatorname{supp} \mu_{\beta,\gamma}, f \in \mathrm{FC}^1.$$

Then B is closable and if B^* is the adjoint of B in $L^2(\mu_{\beta,\gamma})$ then $B^* \supset -B$ so that $\mathrm{i}B$ is symmetric. The Gibbs measures $\mu_{\beta,\gamma}$ are infinitesimally invariant under the Euler flow (in the sense that for any $f \in \mathrm{FC}^1$, $\int_{\mathscr{D}'(\Lambda)} Bf \,\mathrm{d}\mu_{\beta,\gamma} = 0$, or equivalently $B^*1 = B1 = 0$.

Proof. From the definition of B on FC^1 we have for $f \in \mathrm{FC}^1$, $g(\eta) = \tilde{g}(y_{i_1}, \dots, y_{i_k}) \in \mathrm{FC}^1$:

$$(f, Bg) = \sum_{l=1}^{k} \left(f, B_{i_l} \frac{\partial}{\partial y_{i_l}} g\right) = \sum_{l=1}^{k} \left(\left(\frac{\partial}{\partial y_{i_l}}\right)^* B_{i_l} f, g\right), \tag{5.7}$$

since $B_n^*(\eta) = B_n(\eta)$, $B_n(\eta)$ being multiplication by a real-valued function (in $L^2(\mu_\gamma)$). Let us first compute $(\partial/\partial y_n)^*$. From the definition of $\mu_{\beta,\gamma}$ we first get that $\mu_{\beta,\gamma}$ is translation quasi-invariant in any of the directions y_n and

$$\frac{\mathrm{d}\mu_{\beta,\gamma}(\eta + t y_n)}{\mathrm{d}\mu_{\beta,\gamma}(\eta)} = \exp[-\tfrac{1}{2} t^2 (\gamma + \beta \lambda_n^{-1}) y_n^2 - t(\gamma + \beta \lambda_n^{-1}) y_n]. \tag{5.8}$$

But for any $f, g \in \mathrm{FC}^1$,

$$\int \bar{f}(\eta)\frac{\partial}{\partial y_n} g(\eta)\, \mathrm{d}\mu_{\beta,\gamma}(\eta) = \frac{\mathrm{d}}{\mathrm{d}t}\int \bar{f}(\eta) g(\eta + t\eta_n)\, \mathrm{d}\mu_{\beta,\gamma}(\eta)\bigg|_{t=0}. \tag{5.9}$$

On the other hand, by (5.8),

$$\int \bar{f}(\eta) g(\eta + t\eta_n)\, \mathrm{d}\mu_{\beta,\gamma}(\eta)$$

$$= \int \bar{f}(\eta - ty_n) g(\eta) \exp[-\tfrac{1}{2}t^2(\gamma + \beta\lambda_n^{-1})y_n^2 + t(\gamma + \beta\lambda_n^{-1})y_n]\, \mathrm{d}\mu_{\beta,\gamma}(\eta). \tag{5.10}$$

From this it follows that (5.9) is equal to

$$\int -\frac{\partial \bar{f}}{\partial y_n}(\eta) g(\eta)\, \mathrm{d}\mu_{\beta,\gamma}(\eta) - (\gamma + \beta\lambda_n^{-1}) \int y_n \bar{f}(\eta) g(\eta)\, \mathrm{d}\mu_{\beta,\gamma}(\eta). \tag{5.11}$$

This proves

$$\left(\frac{\partial}{\partial y_n}\right)^* \supset -\frac{\partial}{\partial y_n} + (\gamma + \beta\lambda_n^{-1}) y_n. \tag{5.12}$$

From (5.12) and (5.7) we see that

$$B^* \supset -\sum_n B_n(\eta)\frac{\partial}{\partial y_n} + \sum_n (\gamma + \beta\lambda_n^{-1}) y_n B_n(\eta) \tag{5.13}$$

and

$$(f, Bg) = -\sum_{l=1}^{k}\left(B_{i_l}(\eta)\frac{\partial}{\partial y_{i_l}} f, g\right) + \sum_{l=1}^{k}((\gamma + \beta\lambda_{i_l}^{-1}) y_{i_l} B_{i_l}(\eta) f, g).$$

If we prove that

$$\sum_{l=1}^{k} (\gamma + \beta\lambda_{i_l}^{-1}) y_{i_l} B_{i_l}(\eta) f = 0,$$

we get $B^* \restriction \mathrm{FC}^1 = B \restriction \mathrm{FC}^1$. This is proven in the following lemma.

Lemma 5.2. $\sum_{n=1}^{k} (\gamma + \beta\lambda_n^{-1}) y_n B_n(\eta) = 0, \quad \mu_{\beta,\gamma}$-*a.s.*

Proof. Let us replace η by $\tilde{\eta}_N = \sum_{\lambda_n \le N} \varphi_n y_n$. Let $y_n(t)$ be the classical solution of the Euler equation

$$\frac{\partial}{\partial t} y_n(t) = B_n(\tilde{\eta}_N(t)), \quad \text{with } \tilde{\eta}_N(t) \equiv \sum_{\lambda_n \le N} \varphi_n y_n(t).$$

Then $S(\tilde{\eta}_N)$ and $H(\tilde{u}_N)$, with rot $\tilde{u}_N = \tilde{\eta}_N$ exist and are independent of t. Thus

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} S(\tilde{\eta}_N) = \sum_n \frac{\partial}{\partial y_n} S(\tilde{\eta}_N) B_n(\tilde{\eta}_N(t)) = \sum_n y_n(t) B_n(\tilde{\eta}_N(t)),$$

and hence in particular $\sum_n y_n B_n(\tilde{\eta}_N) = 0$.

Similarly,

$$0=\frac{\mathrm{d}}{\mathrm{d}t}H(\tilde{u}_N)=\sum_n \lambda_n^{-1}y_n(t)B_n(\tilde{\eta}_N(t)),$$

where we used $(\partial/\partial y_n)H(\tilde{u}_N)=\lambda_n^{-1}y_n$. Taking this for $t=0$ we see that the lemma is proven when $B_n(\eta)$ is replaced by $B_n(\tilde{\eta}_N)$. But $B_n(\tilde{\eta}_N)\to B_n(\eta)$ strongly as $N\to\infty$ in $L^2(\mu_\gamma)$. Hence there is a subsequence converging μ_γ-a.s.

Using this subsequence we get

$$0=\sum_{n=1}^{k}(\gamma+\beta\lambda_n^{-1})y_nB_n(\tilde{\eta}_N)\to\sum_{n=1}^{k}(\gamma+\beta\lambda_n^{-1})y_nB_n(\eta),\quad \mu_\gamma\text{-a.s.,}\quad \text{as } N\to\infty.$$

Thus

$$\sum_{n=1}^{k}(\gamma+\beta\lambda_n^{-1})y_nB_n(\eta)=0,\quad \mu_\gamma\text{-a.s.}$$

This proves the lemma and the theorem. □

We shall now remark that B can actually be extended linearly to the larger domain $D(B)=P\cup \mathrm{FC}^1$, with P defined in (3.12), since for any $f\in P\cup\mathrm{FC}^1$ we have

$$(Bf)(\eta)=\sum_n B_n(\eta)\frac{\partial}{\partial y_n}f(\eta)\in L^2(\mu_\gamma).$$

Since $:H:_\gamma\in P_2$ we have then $:H:_\gamma\in D(B)$.

By the infinitesimal invariance of $\mu_{\beta,\gamma}$ we get easily $B{:}H{:}_\gamma=0$. Let us now assume that $\alpha=0$, which is the case when Λ is a bounded domain with $\partial\Lambda\in C^1$ and consisting of only one connected component. In this case define an antilinear map J by

$$Jf(\eta)=\bar{f}(-\eta)$$

for all $f\in L^2(\mu_\gamma)$. Then $J^2=1$ and J is a complex conjugation in the complex Hilbert space $L^2(\mu_\gamma)$. Since $\alpha=0$ we have from the definition of $B_n(\eta)$ that

$$B_n(-\eta)=B_n(\eta).$$

This implies, with $A=\mathrm{i}B$,

$$JA=AJ,$$

which shows that A is real with respect to J. In the preceding theorem we showed that A is symmetric in $L^2(\mu_\gamma)$. By a general result of von Neumann, A being real with respect of the conjugation J and symmetric, there exists at least one self-adjoint extension $\tilde{A}$ of A s.t.

$$J\tilde{A}=\tilde{A}J$$

and $\tilde{A}^*=\tilde{A}$.

Setting then $\tilde{B} = -i\tilde{A}$ we have $\tilde{B} \supset B$,

$$J\tilde{B} = -\tilde{B}J$$

and $(\tilde{B})^* = -\tilde{B}$. Then $e^{t\tilde{B}}$, $t \in \mathbb{R}$ is a strongly continuous unitary group on $L^2(\mu_\gamma)$. Hence we have proven the following theorem.

Theorem 5.3. *If $\alpha = 0$ then the operator $B = \sum_n B_n(\partial/\partial y_n)$ (infinitesimal generator of the generalized Euler flow) is such that iB is symmetric in $L^2(\mu_{\beta,\gamma})$ and is real with respect to the conjugation $Jf(\eta) = \bar{f}(-\eta)$. Also, iB has a self-adjoint extension $i\tilde{B}$, which generates a strongly continuous unitary group $U_t = e^{t\tilde{B}}$, $t \in \mathbb{R}$, on $L^2(\mu_{\beta,\gamma})$, such that for any $f \in D(B)$*

$$\text{s-}\lim_{t \to 0} \frac{1}{t}(U_t f - f) = Bf,$$

the limit being in the strong $L^2(\mu_{\beta,\gamma})$-sense.

Remark. We can look upon U_t as defining a Koopman–Kolmogorov "generalized flow", in the $L^2(\mu_{\beta,\gamma})$-sense, associated with the Euler equation.

Let us now come back to the general case $\alpha \neq 0$. If A is essentially self-adjoint then there is only a self-adjoint extension $\tilde{A}$ of A and $\tilde{A} = A^*$. Hence in this case there is only one extension $\tilde{B}$ of B s.t. $\tilde{B}^* = -\tilde{B}$, and the extension $\tilde{B}$ is just the closure $\bar{B}$ of B.

$e^{t\bar{B}}$ is then a strongly continuous unitary group in $L^2(\mu_{\beta,\gamma})$. B being a derivation, we can associate to it an automorphism of $L^\infty(\mu_{\beta,\gamma})$, implemented by a μ_γ-measurable transformation ϕ_t of $\mathcal{D}'(\Lambda)$, "a generalized Euler flow" supported by μ_γ, s.t. $e^{t\bar{B}}f(\eta) = f(\phi_t(\eta))$ for μ_γ-a.e. η and all $f \in L^2(\mu_{\beta,\gamma})$. Of course in this case $\mu_{\beta,\gamma}$ is invariant under ϕ_t and for $f \in \mathrm{FC}^1$ we have

$$\frac{d}{dt} f(\phi_t(\eta))\Big|_{t=0} = Bf(\eta), \quad \mu_{\beta,\gamma}\text{-a.s.}$$

Moreover the μ_γ-measurable function $:H:_\gamma = \frac{1}{2}\sum_{n=0}^\infty \lambda_n^{-1}(y_n^2 - \gamma^{-1})$ of Theorem 4.1 is invariant under ϕ_t, since $\mu_{\beta,\gamma}$ is a constant times $e^{-\beta :H:_\gamma}$ times the invariant measure μ_γ.

We summarize these results in the following theorem.

Theorem 5.4. *Let Λ be a bounded domain with $\partial\Lambda \in C^1$. Let B as in Theorem 5.1, then $A = iB$ is symmetric in $L^2(\mu_{\beta,\gamma})$. If A is essentially self-adjoint then its closure $\bar{A}$ is self-adjoint, $\bar{A} = A^*$, and the closure $\bar{B}$ of B satisfies $\bar{B}^* = -B$. $e^{t\bar{B}}$, $t \in \mathbb{R}$, is then a strongly continuous unitary group on $L^2(\mu_{\beta,\gamma})$ and*

$$e^{t\bar{B}}f(\eta) = f(\phi_t(\eta))$$

for $\mu_{\beta,\gamma}$-a.e. η, where ϕ_t is a $\mu_{\beta,\gamma}$-preserving transformation of supp $\mu_\gamma = \mathcal{D}'(\Lambda)$. *For* *any* $f \in \mathrm{FC}^1$ *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\phi_t(\eta))\bigg|_{t=0} = Bf(\eta), \quad \mu_{\beta,\gamma}\text{-a.s.}$$

The renormalized energy $:H:_\gamma \equiv \frac{1}{2}\sum_n \lambda_n^{-1}(y_n^2 - \gamma^{-1})$ *is invariant under* ϕ_t.

Remark. We have not been able to verify the assumption that $\mathrm{i}B$ is essentially self-adjoint on FC^1. For systems of infinitely many interacting particles a corresponding statement has been proven in [18].

The flow ϕ_t can be looked upon as providing a generalized solution $\eta(t) = \phi_t(\eta(0))$ of the Euler equation given by $u(t) = \nabla^\perp(\Delta^{-1}\eta(t) + \alpha)$, for all initial conditions $u(0) = \nabla^\perp(\Delta^{-1}\eta(0) + \alpha)$, $\eta(0) \in \operatorname{supp} \mu_\gamma$. These solutions have infinite energy $H(u)$, for μ_γ-almost all initial conditions, as we have seen in Section 4.

Remark. In [22, 23] a "generalized Euler flow" γ_t associated with the Euler equation in the situation of Theorem 5.3 has been constructed, using a compactness argument starting from the finite dimensional flows corresponding to the cut-off Euler equations of the form of the one discussed in the proof of the Lemma 5.2.[2] In this case if $A = \mathrm{i}B$ is essentially self-adjoint, ϕ_t has to coincide with γ_t. We shall here give the main result of [22, 23], referring to those references for the proof.

Theorem 5.5. *Let* Λ *be a bounded domain of* $\mathbb{R}^2$ *with* $\partial\Lambda$ *connected and* C^1. *Let*

$$B_n^N(\eta) \equiv B_n(\tilde\eta_N) \quad \text{and} \quad B^N \equiv \sum_{\lambda_n \le N} B_n^N(\eta)\frac{\partial}{\partial y_n}.$$

Define $y_n^N(t)$ *as the solution of the cut-off Euler equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} y_n^N(t) = B_n^N(y_m^N(t), \lambda_m \le N),$$

with initial condition $y_n^N(0) = y_n$. *Set*

$$\gamma_t^N(y_n^N(0)) \equiv y_n^N(t), \qquad (U_t^N F)(\eta) \equiv F(\gamma_t^N(\eta)), \quad \eta \in \operatorname{supp} \mu_{\beta,\gamma}.$$

Then U_t^N *is unitary in* $L^2(\mu_{\beta,\gamma})$ *and* $\mu_{\beta,\gamma}$ *is invariant under* γ_t^N.

The family $\{\mathrm{e}^{\mathrm{i}\alpha(\gamma_t^N(y_n))},\ \alpha \in \mathbb{R},\ n \in \mathbb{N}\}$ *indexed by N is equicontinuous, uniformly bounded by* 1, *from* $\mathbb{R}$ *into* $L^2(\mu_{\beta,\gamma})$. *There is a subsequence of N such that* $\mathrm{e}^{\mathrm{i}\alpha(\gamma_t^N(y_n))}$ *converges weakly in* $L^2(\mu_{\beta,\gamma})$, *in a uniform way for all t in compacts of* $\mathbb{R}$, *to a limit* $\tilde f_\alpha(t,\eta)$ *in* $L^2(\mu_{\beta,\gamma})$. *Also,* $\tilde f_\alpha(t,\eta)$ *solves the limit Euler equation in the sense that*

$$\tilde f_\alpha(t) = \tilde f_\alpha(0) + \operatorname*{w\text{-}lim}_{N\to\infty} \int_0^t \tilde B^N \tilde f_\alpha(\gamma_s^N(\cdot))\,\mathrm{d}s,$$

$y_n^N(t)$ *converges weakly by subsequences in* $L^2(\mu_{\beta,\gamma})$ *to a limit* $y_n(t)$ *and one has*

$$\frac{\partial}{\partial\alpha}\tilde f_\alpha(t)\bigg|_{\alpha=0} = \mathrm{i}y_n(t), \quad \text{weakly in } L^2(\mu_{\beta,\gamma}).$$

[2] A corresponding method has been used in [24] to get a similar result in the case of the Euler equation in a box with periodic boundary conditions.

Then, $U_t\tilde{f}_\alpha(s) \equiv \tilde{f}_\alpha(s+t)$, $s, t \in \mathbb{R}$ is a strongly continuous one-parameter group of contractions of $L^2(\mu_{\beta,\gamma})$, with infinitesimal generator $\hat{B}$ an extension of B. $e^{t\hat{B}}$, $(e^{t\hat{B}})^*$ are contraction semigroups in $L^2(\mu_{\beta,\gamma})$ and $\hat{B}$ is maximally densely defined dissipative and there exists a unique orthogonal decomposition $L^2(\mu_{\beta,\gamma}) = \mathcal{H}_1 \oplus \mathcal{H}_2$, with *$U_t \restriction \mathcal{H}_1$ unitary and $U_t \restriction \mathcal{H}_2$ completely non unitary. The maps $\gamma_t(y_n(s)) \equiv y_n(s+t)$ form a weakly continuous and differentiable group from $\mathbb{R}$ into $L^2(\mu_{\beta,\gamma})$.*

Remark. In [19] Hopf considered stochastic flows in connection with the Navier–Stokes equation in a domain Λ. If μ_t is the probability distribution for a stochastic flow u_t, then Hopf considered the corresponding positive definite function $\Psi_t(v)$, defined on the space of smooth vector-valued functions $v \in C^\infty(\Lambda, \mathbb{R}^3)$, given by

$$\Psi_t(v) \equiv \int e^{i\int_\Lambda u(x)v(x)\,dx}\, d\mu_t(u). \tag{5.7}$$

The Hopf equation [19] is obtained by writing the Navier–Stokes equation in terms of the characteristic function Ψ_t. The inviscid Euler–Hopf equation is obtained from the general Hopf equation in [19] by setting the viscosity equal to zero.

Let $u = \nabla^\perp(\Delta^{-1}\eta + \alpha)$, with $\eta \in \operatorname{supp} \mu_{\beta,\gamma}$. Then $\exp(i\int_\Lambda u(x)v(x)\,dx) \in FC^1 \subset D(B)$ and, by the infinitesimal invariance of $\mu_{\beta,\gamma}$ (Theorem 5.1), we have

$$\int (B\, e^{i\int_\Lambda u(x)v(x)\,dx})\, d\mu_{\beta,\gamma}(\eta) = 0. \tag{5.8}$$

Let

$$\Psi_0(v) = \int e^{i\int_\Lambda u(x)v(x)\,dx}\, d\mu_{\beta,\gamma}(\eta).$$

As in the remark at the end of Section 2 we have

$$B\, e^{i\int_\Lambda u(x)v(x)\,dx} = -\int_\Lambda \sum_{i=1}^{2} \frac{\delta\, e^{i\int_\Lambda u(x)v(x)\,dx}}{\delta u_i(x)} ((u\cdot\nabla)u_i(x) + f(x))\,dx. \tag{5.9}$$

The Hopf equation for measures μ^t associated with the Euler equation is by definition the equation

$$\frac{\partial}{\partial t}\Psi_t(v) = \int B(e^{i\int_\Lambda u(x)v(x)\,dx})\, d\mu^t(u). \tag{5.10}$$

Inserting (5.9) into this and expressing the right hand side by functional derivatives of $\Psi_t(v)$ we get the Hopf equation in the form

$$\frac{\partial}{\partial t}\Psi_t(v) = \int_\Lambda \sum_{k=1}^{2} v_k(x)\left[i \sum_{j=1}^{2} \frac{\partial}{\partial x_j} \frac{\delta^2 \Psi_t}{\delta v_j(x)\,\delta v_k(x)} - f_k\right] dx. \tag{5.11}$$

Let $\mu^t = \mu^t_{\beta,\gamma}$, where $\mu^t_{\beta,\gamma}$ is the image of $\mu_{\beta,\gamma}$ under the flow ϕ_t of Theorem 5.4. Then $\mu^t = \mu^0 = \mu_{\beta,\gamma}$ for all t, by the invariance of $\mu_{\beta,\gamma}$ under ϕ_t. Moreover by (5.8) the right hand side of (5.10) is zero, hence the right hand side of (5.11) is zero. We see thus that in this case $\Psi_t(v) = \Psi_0(v)$ is a stationary solution of the Hopf equation.[3]

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Notes added in proof

In deep sorrows the first author announces the departure of his great friend and coauthor for nearly twenty years, Raphael Høegh-Krohn.

Recently the global invariance of the measures $\mu_{\beta,\gamma}$ under a pointwise flow associated with Euler equation has been proven in S. Albeverio, A. B. Cruzeiro, Global flows with invariant (Gibbs) measures for Euler and Navier-Stokes two dimensional fluids, to appear in Comm. Math. Phys.

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